INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2022-23
Statistics - III, Backpaper Examination, December, 2023
Time: 2 Hours **Total Marks: 50**

1. Let $\mathbf{Y} \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Find the conditional distribution of $\mathbf{Y}'\mathbf{Y}$ given $\mathbf{a}'\mathbf{Y} = 0$ where $\mathbf{a}$ is a non-zero constant vector. [8]

2. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n\times p}$ has $\mathbf{1}$ as its first column and rank $r \leq p$, and $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$.
(a) If $\hat{\beta}$ is the least squares estimator of β, show that $(\hat{\beta} - \beta)'\mathbf{X}'\mathbf{X}(\hat{\beta} - \beta)$ is distributed independently of the residual sum of squares.
(b) Find the maximum likelihood estimator of σ^2. Is it unbiased?
(c) Explain how the coefficient of determination, R^2, can be used to check the quality of the fitted linear model. $[6 + 6 + 6]$

3. Consider the following model:
$y_1 = \theta + \gamma + \epsilon_1$
$y_2 = \theta + \phi + \epsilon_2$
$y_3 = 2\theta + \phi + \gamma + \epsilon_3$
$y_4 = \phi - \gamma + \epsilon_4$,
where ϵ_i are uncorrelated having mean 0 and variance σ^2.
(a) Show that $\gamma - \phi$ is estimable. What is its BLUE?
(b) Find the residual sum of squares. What is its degrees of freedom?$[8 + 6]$

4. Let Y be a response variable and $X_1, \ldots, X_k$ be covariates. Also, let ρ_i denote the correlation coefficient between Y and X_i, and let R denote the multiple correlation coefficient between Y and $X_1, \ldots, X_k$.
(a) Show that $R \geq \max\{|r_i|, 1 \leq i \leq k\}$.
(b) What is the exact relationship between R and r_i's when $k = 1$? [5+5]